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The quantum Schrödinger equation and the q-deformation of the hydrogen atom*

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Abstract. By employing the non-commutative differential calculus on the quantum orthogonal planes, we investigate the deformed version of the Schrödinger equation for the attractive Coulomb potential $V(r)=-e^2/r$. The solutions show the quantum group symmetry, and the calculated energy eigenvalues seem to be proportional to $[n]^{-2}$, which is in agreement with the energy levels of the hydrogen atom in the limit of $q\to 1$.

1. Introduction

Recently, much attention has been paid to the so-called quantum groups, or the quantum universal enveloping algebras [1], which originated from the research of trigonometric or hyperbolic solutions of the quantum Yang-Baxter equations [2]. It has been shown that they are closely related to various physically interesting models and theories, such as the exactly soluble statistical models [2], inverse scattering method for nonlinear evolution equation [3], factorizable S matrix and integrable field theory [4], conformal field theory and topological Chern-Simons theory [5].

The mathematical structure of the quantum enveloping algebras has been systematically carried out by Drinfeld [1], Jimbo [1] and Reshetikhin *et al* [6]. On the other hand, Woronowicz [1] and Manin [7] use another way to approach the subject, which in a sense is more intriguing to physicists. Woronowicz defines a consistent differential calculus on the non-commutative space of a quantum group and therefore makes quantum groups a concrete example of non-commutative differential geometry [8]. Manin considers a quantum group as effecting linear transformations upon a quantum plane, whose coordinates belong to a non-commutative associative algebra.

More recently, Wess and Zumino [9], by applying Woronowicz's differential geometry method to the quantum plane and interpreting Manin's dual space of the quantum plane as differentials of the coordinates, give a simpler example of non-commutative differential geometry. As stressed by Wess and Zumino, the differential calculus is covariant with respect to the action of the quantum group $GL_q(n)$, and the nilpotent condition $d^2=0$ does not contradict other consistency requirements. In this

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paper, we will first present Wess–Zumino's differential calculus on the quantum plane, with the emphasis that some of the consistency conditions have 'gauge' freedom, which is very useful in the following discussion. We also point out that the condition $d^2=0$ is not only consistent with other requirements but also a direct consequence of the construction method adopted here. Furthermore Wess–Zumino's solution to the consistency conditions is generalized to include much more general cases, e.g., the case of the R-matrix satisfying B_n type solutions of the Yang–Baxter equation. This kind of solution has already been considered in [10, 11]. We give a concrete example for the three-dimensional case in section 3. In an attempt to find the possible connection of this q-space to our real physical space, we consider the Schrödinger-like equation, concretely the one with the attractive Coulomb potential. The results of the calculation demonstrate that the system has a close relation to the hydrogen atom, about which a brief discussion is given at the end of the paper.

2. Non-commutative differential calculus on the quantum plane

Manin's quantum plane is defined as follows [7]: Variables x^i, $i=1,2,\ldots,n$, belonging to a non-commutative associative algebra, satisfy the commutation relations

$$x^i x^j - B^{ij}_{kl} x^k x^l = 0. \tag{2.1}$$

Using the standard tensor product notation, the relations (2.1) read as

$$(E_{12}-B_{12})x_1 x_2 = 0 \tag{2.1'}$$

where E is the unit matrix. We introduce the exterior differential d as follows

$$d = \xi^i \partial_i \tag{2.2}$$

where ξ^i differential and ∂_i derivatives satisfying

$$\partial_i x^j = \delta^j_i. \tag{2.3}$$

As usual, the exterior differential d is required to obey the following conditions

(i) nilpotent

$$d^2 = 0 \tag{2.4}$$

(ii) Leibniz rule

$$d(fg) = (df)g + (-)^{\tilde{f}} f(dg) \tag{2.5}$$

where $\tilde{f}$ is the degree of the 'form' f. Functions of the variables x^i are defined as formal power series while forms power series of both variables x and differentials ξ. Relations (2.1) are sufficient to order in some standard way an arbitrary monomial of x. But one also needs the commutation relations between ξ and x and between two ξ's in ordering variables and differentials within a form. In general they are not commuting with each other and the relations can be written as

$$f\xi^j = (-1)^{\tilde{f}} \xi^k O^j_k f \tag{2.6}$$

with O^j_k an operator acting upon f. The derivatives do not satisfy the simple Leibniz rule of commutative algebra. Indeed we have

$$\partial_k(fg) = (\partial_k f)_g + (O^j_k f)\partial_j g \tag{2.7}$$

from (2.2) and (2.5).

Now for the quantum plane the operator O^j_k is linear. Suppose

$$O^j_k x^i = C^{ij}_{kl} x^l \qquad \text{for } f = x^i \tag{2.8a}$$

$$O^j_k \xi^i = D^{ij}_{kl} \xi^l \qquad \text{for } f = \xi^i \tag{2.8b}$$

with C and D being numerical matrices to be specified later. This gives

$$x^i \xi^j = C^{ij}_{kl} \xi^k x^l \qquad \text{or} \qquad x_1 \xi_2 = C_{12} \xi_1 x_2 \tag{2.9}$$

and

$$\xi^i \xi^j = -D^{ij}_{kl} \xi^k \xi^l \qquad \text{or} \qquad (E_{12} + D_{12}) \xi_1 \xi_2 = 0 \tag{2.10}$$

from (2.6) and

$$\partial_k x^i = \delta^i_k + C^{ij}_{kl} x^l \partial_j \tag{2.11}$$

$$\partial_k \xi^i = D^{ij}_{kl} \xi^l \partial_j \tag{2.12}$$

from (2.7). To complete the exterior algebra, we still need the commutation relations among derivatives

$$\partial_i \partial_j - F^{lk}_{ji} \partial_k \partial_l = 0 \qquad \text{or} \qquad \partial_2 \partial_1 - \partial_2 \partial_1 F_{12} = 0 \tag{2.13}$$

where F is yet another matrix to be determined.

The consistency conditions of the differential calculus, first proposed by Wess and Zumino, can be summed up as

$$(E_{12} - B_{12})(E_{12} + C_{12}) = 0 \tag{2.14}$$

$$(E_{12} - B_{12}) C_{23} C_{12} = C_{23} C_{12} (E_{23} - B_{23}) \tag{2.15}$$

$$D = C^{-1} \tag{2.16}$$

and

$$D_{23} C_{12} C_{23} = C_{12} C_{23} D_{12} \tag{2.17}$$

together with another two equations similar to (2.14), (2.15), i.e.

$$(E_{12} - F_{12})(E_{12} + C_{12}) = 0 \tag{2.18}$$

and

$$(E_{23} - F_{23}) C_{12} C_{23} = C_{12} C_{23} (E_{12} - F_{12}). \tag{2.19}$$

Equations (2.14)-(2.19) complete all the consistency requirements for a consistent definition of differential calculus on a quantum plane. In establishing these consistency conditions the only essential assumptions are the Leibniz rule (2.5) and the nilpotent acting upon the single x, $d^2 x = 0$.

Now we give some remarks about the properties of the matrices B, C, D and F. Firstly consider relation (2.1). It is obvious that $B^2, B^3, \ldots, B^n, \ldots$ should give out the same commutation relation for x's as B does. More generally, any function $\phi(B)$ of B with

$$\phi(E) = E$$

can be used in place of B. In fact, we have

$$E - \phi(B) = \phi_1(B)(E - B). \tag{2.20}$$

Considering the fact that $(E-B)$ always appears as a whole in the consistency conditions, one sees that, if B_0 is an appropriate candidate for B matrix in (2.1), $\phi(B_0)$ is another equally qualified choice. All the choices with different function ϕ give the same relation among variables x's. We call this a 'gauge' freedom for B. In the case where $E-B_0$ is a projection operator, as is widely used later, the ϕ_1 factor in (2.20) is just a numerical coefficient. The same argument is also true for matrix F. As a matter of fact, we can choose F identifying to B, since in all the consistency conditions F appears just as same as B does.

Similarly for D matrix in (2.10), $-D$, D^2, $-D^3, \ldots, (-D)^n, \ldots$ should give the same commutation relations for ξ's. A similar 'gauge' freedom exists for D matrix, i.e. $\psi(D)$ with

$$\psi(-E) = -E$$

can be used instead of D

$$E + \psi(D) = \psi_1(D)(E+D). \tag{2.21}$$

A special choice $\psi_1(D) = D^{-1} = C$ yields $E + \psi(D) = E + C$ which is compatible with the result by applying the differential d acting upon (2.9). Notice that C appears in the consistency conditions sometimes by itself and sometimes in the combination $E+C$. So one must choose a fixed gauge to make C meets other requirements as in (2.15), (2.17) and (2.19).

The simplest solutions to the consistency conditions (2.14)-(2.19) have been obtained by Wess and Zumino [9]. These solutions are closely related to A_n-type R-matrices satisfying the Yang–Baxter equation in the braid form

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{2.22}$$

and having two different eigenvalues

$$(\check{R}-\lambda_1)(\check{R}-\lambda_2) = 0 \tag{2.23}$$

with $\lambda_2 = q$ corresponding to the symmetric combination and $\lambda_1 = -q^{-1}$ the antisymmetric in the $q \to 1$ limit. Correspondingly, there are two projection operators,

$$Q^{(2)} = Q_S = \frac{E + q\check{R}}{1+q^2} \qquad Q^{(1)} = Q_A = \frac{E - q^{-1}\check{R}}{1+q^{-2}} \tag{2.24}$$

satisfying

$$Q^{(\alpha)}Q^{(\beta)} = \delta^{\alpha\beta}Q^{(\beta)} \qquad Q^{(1)} + Q^{(2)} = E. \tag{2.25}$$

Condition (2.14) tells us that two combinations $(E-B)$ and $(E+C)$ are orthogonal, so we can choose

$$E - B = bQ^{(1)} \qquad E + C = cQ^{(2)}. \tag{2.26}$$

We can further choose the constant b to give a simpler form of B

$$B = q^{-1}\check{R} \tag{2.27}$$

whereas c must be determined by the requirement of C satisfying the YB-type relation (2.15), (2.17) and (2.19). There are two possible choices:

$$c = 1+q^2 \qquad C = q\check{R} \tag{2.28a}$$

$$c = 1+q^{-2} \qquad C = q^{-1}\check{R}^{-1}. \tag{2.28b}$$

Together with the identification $F = B$ and $D = C^{-1}$, all the consistency conditions (2.14)-(2.19) are fulfilled.

The other choice

$$E - B \propto Q^{(2)} \qquad E + C \propto Q^{(1)}$$

will give fermionic relations among x's and bosonic relations among ζ's.

Another important relation directly results from the above construction and (2.12)

$$\begin{aligned} d^2 &= \xi^i \partial_i \xi^j \partial_j = \xi^i (D^{jk}_{il} \xi^l \partial_k) \partial_j \\ &= \xi_1 \xi_2 D_{12} \partial_2 \partial_1 = \xi_1 \xi_2 (q^2 Q^{(2)}_{12} - Q^{(1)}_{12}) \partial_2 \partial_1 = 0 \end{aligned} \tag{2.29}$$

where the D-matrix is decomposed as a sum of two projection terms and use has been made of the relations (2.10) and (2.13) with the identification given in (2.24). Equation (2.29) means that the nilpotent rule holds not only when of acting on x itself, but also in the general sense.

The same method can be used to construct new solutions to the consistency conditions. We first recall that the $\check{R}$ matrices satisfying the braid Yang–Baxter equation have three different eigenvalues for B_n, C_n, and D_n type solutions [1, 12]

$$(\check{R} - \lambda_2)(\check{R} - \lambda_1)(\check{R} - \lambda_0) = 0. \tag{2.30}$$

For B_n-type, we have

$$\lambda_2 = q \qquad \lambda_1 = -q^{-1} \qquad \lambda_0 = q^{-2n} \tag{2.31}$$

which correspond to the quintet, the triplet and the singlet respectively in the B_1 case. Three projection operators can be constructed as follows

$$\begin{aligned} Q^{(2)} &= \frac{(\check{R} - \lambda_0)(\check{R} - \lambda_1)}{(\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1)} \\ Q^{(1)} &= \frac{(\check{R} - \lambda_0)(\check{R} - \lambda_2)}{(\lambda_1 - \lambda_0)(\lambda_1 - \lambda_2)} \\ Q^{(0)} &= \frac{(\check{R} - \lambda_1)(\check{R} - \lambda_2)}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)} \end{aligned} \tag{2.32}$$

with the properties

$$Q^{(\alpha)} Q^{(\beta)} = \delta^{\alpha\beta} Q^{(\beta)} \qquad Q^{(0)} + Q^{(1)} + Q^{(2)} = E. \tag{2.33}$$

All the matrices comprising $\check{R}$ can be re-expressed in terms of projection operators, e.g.,

$$\begin{aligned} \check{R} &= \lambda_2 Q^{(2)} + \lambda_1 Q^{(1)} + \lambda_0 Q^{(0)} \\ \check{R}^{-1} &= \lambda_2^{-1} Q^{(2)} + \lambda_1^{-1} Q^{(1)} + \lambda_0^{-1} Q^{(0)}. \end{aligned} \tag{2.34}$$

As mentioned above, $(E - B)$ are orthogonal to $(E + C)$. We can choose $(E - B)$ as one of the projection operators, say $Q^{(1)}$. Then $(E + C)$ must be the linear combination of the other two projection operators, i.e.

$$E + C = c_0 Q^{(0)} + c_2 Q^{(2)}. \tag{2.35}$$

Now since each of Q's is quadratic in $\check{R}$, we must choose the coefficients c_0, c_2 appropriately to eliminate these quadratic terms in order to ensure C satisfying the Yang–Baxter type conditions (2.15), (2.17) and (2.19). This is achieved by taking

$$c_0 = 1 - \frac{\lambda_0}{\lambda_1} = 1 + q^{-1} \qquad c_2 = 1 - \frac{\lambda_2}{\lambda_1} = 1 + q^2. \tag{2.36}$$

Then

$$C = -\lambda_1^{-1}\check{R} = q\check{R} \qquad D = -\lambda_1 \check{R}^{-1} = q^{-1}\check{R}^{-1}. \tag{2.37}$$

Alternatively we can choose

$$C = -\lambda_1 \check{R}^{-1} = q^{-1}\check{R}^{-1} \qquad D = -\lambda_1^{-1}\check{R} = q\check{R}$$

corresponding to $c_0 = 1 - \lambda_1/\lambda_0 = 1 + q$, $c_2 = 1 - \lambda_1/\lambda_2 = 1 + q^{-2}$. It is not difficult to see from the Yang–Baxter equation (2.22) that

$$\check{R}_{12}^{-1}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}^{-1} \qquad \check{R}_{12}^{2}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}^{2}. \tag{2.38}$$

from which one obtains

$$Q_{12}^{(\alpha)}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}Q_{23}^{(\alpha)} \qquad \check{R}_{12}\check{R}_{23}Q_{12}^{(\alpha)} = Q_{23}^{(\alpha)}\check{R}_{12}\check{R}_{23}. \tag{2.39}$$

Now it is straightforward to show that C, D given in (2.37) and $B = F = E - Q^{(1)}$ meet all the consistency requirements. The commutation rules among x's and ξ's now take the following forms

$$(E_{12} - B_{12})x_1x_2 = 0 \Leftrightarrow Q_{12}^{(1)}x_1x_2 = 0 \tag{2.40}$$

$$\partial_2\partial_1(E_{12} - F_{12}) = 0 \Leftrightarrow \partial_2\partial_1 Q_{12}^{(1)} = 0 \tag{2.41}$$

$$(E_{12} + C_{12})\xi_1\xi_2 = 0 \Leftrightarrow \begin{cases} Q_{12}^{(0)}\xi_1\xi_2 = 0 \\ Q_{12}^{(2)}\xi_1\xi_2 = 0 \end{cases} \tag{2.42}$$

$$x_1\xi_2 = C_{12}\xi_1x_2 \Leftrightarrow \begin{cases} Q_{12}^{(1)}x_1\xi_2 = -Q_{12}^{(1)}\xi_1x_2 \\ Q_{12}^{(0)}x_1\xi_2 = -\lambda_0\lambda_1^{-1}Q_{12}^{(0)}\xi_1x_2 \\ Q_{12}^{(2)}x_1\xi_2 = -\lambda_2\lambda_1^{-1}Q_{12}^{(2)}\xi_1x_2. \end{cases} \tag{2.43}$$

The nilpotent condition for the differential d follows once again from this construction:

$$\begin{aligned} d^2 &= \xi^i\partial_i\xi^j\partial_j = \xi^i(D_{il}^{jk}\xi^l\partial_k)\partial_j \\ &= \xi_1\xi_2D_{12}\partial_2\partial_1 = \xi_1\xi_2(-\lambda_1\lambda_0^{-1}Q^{(0)} - \lambda_1\lambda_2^{-1}Q^{(2)} - Q^{(1)})_{12}\partial_2\partial_1 \\ &= -(\lambda_1\lambda_0^{-1}Q^{(0)} + \lambda_1\lambda_2^{-1}Q^{(2)})_{il}^{jk}\xi^i\xi^l\partial_k\partial_j - \xi^i\xi^l\partial_k\partial_jQ^{(1)}{}_{il}^{jk} = 0 \end{aligned} \tag{2.44}$$

where we have inserted D as in (2.37) and (2.34) and taking into account of the relations in (2.42) and (2.43). In fact the nilpotent property of d is ensured when the constraints among the variables (or derivatives) are complementary to ones among differentials. In this case when D matrix is decomposed as the sum of several projection terms, these terms must be divided into two parts: one part will annihilate the differentials and the other annihilates the derivatives.

As in the A_n-type case [9], one can see immediately that all relations of the differential calculus described above are covariant under the action of a linear transformation

$$x^i \to t_j^i x^j \qquad \xi^i \to t_j^i \xi^j \tag{2.45}$$

provided the matrix $T=\{t^i_j\}$ satisfying

$$T_1 T_2 \check{R}_{12} = \check{R}_{12} T_1 T_2. \tag{2.46}$$

The calculus is covariant under the action of quantum group $U_q(A)$ when $\check{R}$ is the A type solution to the Yang–Baxter equation. The planes now under consideration may be called quantum orthogonal planes ($A = B_n, D_n$) or symplectic planes ($A = C_n$).

It is obvious that the method given here in constructing the solutions to the consistency conditions is a general one, which can be directly applied to the cases with $\check{R}$ possessing more than three eigenvalues. For example, in G_2 case, $\check{R}$ has four different eigenvalues. $(E-B)$ is always chosen as a projection operator associated with the antisymmetric representation in the $q \to 1$ limit. And the $(E+C)$ has to be expressed as linear combination of all the other projection operators. Correctly choosing these combination coefficients, one can get a C-matrix proportional to $\check{R}$ itself. Together with $F = B$ and $D = C^{-1}$, all the consistency conditions are well satisfied. The nilpotent of the differential d follows directly from this construction. For quantum orthogonal planes the results presented here coincide with those of [10].

3. The three-dimensional case

In this section, as an example, we give the commutation relations of the differential calculus on the three-dimensional orthogonal quantum plane. The $\check{R}$ matrix is

$$\check{R} = \begin{array}{c} \\ ++ \\ +0 \\ +- \\ 0+ \\ 00 \\ 0- \\ -+ \\ -0 \\ -- \end{array} \begin{array}{ccccccccc} ++ & +0 & +- & 0+ & 00 & 0- & -+ & -0 & -- \\ q & & & & & & & & \\ & d & & 1 & & & & & \\ & & d(1-q^{-1}) & & -dq^{-1/2} & & q^{-1} & & \\ & 1 & & 0 & & & & & \\ & & -dq^{-1/2} & & 1 & & & & \\ & & & & & d & & 1 & \\ & & q^{-1} & & & & 0 & & \\ & & & & & 1 & & 0 & \\ & & & & & & & & q \end{array} \tag{3.1}$$

where $d = q - q^{-1}$ and the blank space means corresponding entries are zero. We denote the coordinates x^+, x^0 and x^-, the differentials

$$dx^+ = \xi^+ \qquad dx^0 = \xi^0 \qquad dx^- = \xi^- \tag{3.2}$$

and the derivatives

$$\frac{\partial}{\partial x^+} = \partial_+ \qquad \frac{\partial}{\partial x^0} = \partial_0 \qquad \frac{\partial}{\partial x^-} = \partial_-. \tag{3.3}$$

For the B_1 case, it is easy to solve the eigenvalue problem of the $\check{R}$ and categorize the eigenvectors according to their eigenvalue as follows:

$$\check{R}^{ij}_{kl}\Phi^{kl} = \lambda_s \Phi^{ij} \tag{3.4}$$

with the eigenvectors

$$\Phi^{ij} = \begin{cases} v^{ij} & & \text{for singlet } \lambda_0 = q^{-2} \\ u^{ij}_m & m = 1, 0, -1 & \text{for triplet } \lambda_1 = 1 = -q^{-1} \\ w^{ij}_\mu & \mu = 2, 1, 0, -1, -2 & \text{for quintet } \lambda_2 = q. \end{cases} \tag{3.5}$$

It should be pointed out that these eigenvectors are just the CG coefficients and in the eigenvectors of the singlet play the role of metric when we construct particular a Euclidean space from the orthogonal plane. For later convenience we write them here explicitly:

$$v^{(ij)}=(v^{+-},v^{00},v^{-+})=(q^{-1/2},1,q^{1/2})/\sqrt{q^{-1}+1+q} \tag{3.6}$$

$$\begin{aligned} &u_1^{(ij)}=(u_1^{+0},u_1^{0+})=(-q^{-1/2},q^{1/2})/\sqrt{q^{-1}+q}\\ &u_0^{(ij)}=(u_0^{+-},u_0^{00},u_0^{-+})=(1,q^{1/2}-q^{-1/2},-1)/\sqrt{q^{-1}+q}\\ &u_{-1}^{(ij)}=(u_{-1}^{0-},u_{-1}^{-0})=(-q^{-1/2},q^{1/2})/\sqrt{q^{-1}+q}\end{aligned} \tag{3.7}$$

$$\begin{aligned} &w_2^{++}=1 \qquad w_1^{+0}=\sqrt{\frac{q}{q+q^{-1}}} \qquad w_1^{0+}=\sqrt{\frac{q^{-1}}{q+q^{-1}}}\\ &(w_0^{+-},w_0^{00},w_0^{-+})=\frac{(q,-q^{1/2}-q^{-1/2},q^{-1})}{\sqrt{(q+q^{-1})(q+1+q^{-1})}}\\ &w_{-2}^{--}=1 \qquad w_{-1}^{0-}=w_1^{+0} \qquad w_{-1}^{-0}=w_1^{0+}.\end{aligned} \tag{3.8}$$

As usual, the orthonormality of these functions is assumed

$$\Phi_s^{ij}\Phi_{ij}^t=\delta_s^t \qquad \Phi_s^{ij}\Phi_{kl}^s=\delta_k^i\delta_l^j. \tag{3.9}$$

The projection operators can be expressed by their eigenvectors as follows

$$Q^{(0)}{}_{kl}^{ij}=v^{ij}v_{kl} \qquad Q^{(1)}{}_{kl}^{ij}=u_m^{ij}u_{kl}^m \qquad Q^{(2)}{}_{kl}^{ij}=w_\mu^{ij}w_{kl}^\mu. \tag{3.10}$$

This coincides with the expressions resulting from (2.32).

Then we have the commutation relations between the coordinates

$$x^+x^0=qx^0x^+ \qquad x^0x^-=qx^-x^0 \qquad x^+x^--x^-x^+=(q^{-1/2}-q^{1/2})x^0x^0 \tag{3.11}$$

the relations between the differentials

$$\begin{aligned} &\xi^+\xi^+=0 \qquad \xi^-\xi^-=0 \qquad \xi^+\xi^0=-q^{-1}\xi^0\xi^-\\ &\xi^0\xi^-=-q^{-1}\xi^-\xi^0 \qquad \xi^+\xi^-=-\xi^-\xi^+ \qquad \xi^0\xi^0=(q^{-1/2}-q^{1/2})\xi^-\xi^+\end{aligned} \tag{3.12}$$

and the relations between the derivatives

$$\partial_+\partial_0=q^{-1}\partial_0\partial_+ \qquad \partial_+\partial_--\partial_-\partial_+=(q^{1/2}-q^{-1/2})\partial_0\partial_0 \qquad \partial_0\partial_-=q^{-1}\partial_-\partial_0. \tag{3.13}$$

We also give the commutation relations between the coordinates and differentials for $C=q\check{R}$,

$$\begin{aligned} &x^+\xi^+=q^2\xi^+x^+ \qquad x^+\xi^0=q\xi^0x^++(q^2-1)\xi^+x^0\\ &x^+\xi^-=\xi^-x^++(q^{-1}-q)q^{1/2}\xi^0x^0-(q^{-1}-q)(q-1)\xi^+x^-\\ &x^0\xi^+=q\xi^+x^0 \qquad x^0\xi^0=q\xi^0x^0+(q^{-1}-q)q^{1/2}\xi^+x^-\\ &x^0\xi^-=q\xi^-x^0+(q^2-1)\xi^0x^-\\ &x^-\xi^+=\xi^+x^- \qquad x^-\xi^0=q\xi^0x^- \qquad x^-\xi^-=q^2\xi^-x^-\end{aligned} \tag{3.14}$$

and the relations between the coordinates and the derivatives

$$\begin{aligned} &\partial_+x^+=1-(q^{-1}-q)(q-1)x^-\partial_-+(q^2-1)x^0\partial_0+q^2x^+\partial_+\\ &\partial_+x^0=(q^{-1/2}-q^{3/2})x^-\partial_0+qx^0\partial_+ \qquad \partial_+x^-=x^-\partial_+\\ &\partial_0x^+=(q^{-1/2}-q^{3/2})x^0\partial_-+qx^+\partial_0 \qquad \partial_0x^0=1+(q^2-1)x^-\partial_-+qx^0\partial_0\\ &\partial_0x^-=qx^-\partial_0 \qquad \partial_-x^+=x^+\partial_- \qquad \partial_-x^0=qx^0\partial_- \qquad \partial_-x^-=1+q^2x^-\partial_-\end{aligned} \tag{3.15}$$

and those between the derivatives and the differentials

$$\begin{aligned}
&\partial_+\xi^+=q^{-2}\xi^+\partial_+ \qquad \partial_+\xi^0=q^{-1}\xi^0\partial_+ \qquad \partial_+\xi^-=\xi^-\partial_+\\
&\partial_0\xi^+=q^{-1}\xi^+\partial_0 \qquad \partial_0\xi^0=q^{-1}\xi^0\partial_0+(q^{-2}-1)\xi^+\partial\\
&\partial_0\xi^-=q^{-1}\xi^{-1}\partial_0+(q^{1/2}-q^{-3/2})\xi^0\partial_+ \qquad \partial_-\xi^+=\xi^+\partial_-\\
&\partial_-\xi^0=q^{-1}\xi^0\partial_-+(q^{1/2}-q^{-3/2})\xi^+\partial_0\\
&\partial_-\xi^-=q^{-2}\xi^-\partial_-+(q^{-2}-1)\xi^0\partial_0+(q^{-1}-q)(q^{-1}-1)\xi^+\partial_+.
\end{aligned} \tag{3.16}$$

As pointed out in the last section, the differential calculus is covariant under the action of the linear transformation

$$x \to Tx \qquad \xi \to T\xi.$$

The singlet combination is invariant under the action of the quantum group B_1:

$$r^2 \equiv g_{ij}x^ix^j=q^{-1/2}x^+x^-+x^0x^0+q^{1/2}x^-x^+ \tag{3.17}$$

where

$$g_{ij}=\sqrt{q+1+q^{-1}}\,v_{ij}.$$

This is nothing but the q-sphere condition. So the quantum orthogonal plane is fundamentally related to the quantum sphere [13]. Furthermore, we see from repeatedly using (3.11) that

$$r^2x^i=x^ir^2. \tag{3.18}$$

This means that, $r^2=xgx=x^ig_{ij}x^j$ is the centre of the algebra generated by the coordinates x.

4. The Schrödinger equation and the q-hydrogen atom

As an application to the differential calculus presented above, we consider the Schrödinger equation for the attractive Coulomb potential $V(r)=-e^2/r$ in the three-dimensional quantum Euclidean space. The results apply to higher-dimensional space.

As stressed in section 3 there exists a metric g for the quantum plane

$$g_{ij}\equiv\sqrt{[3]_{q^{1/2}}}\,v_{ij} \tag{4.1}$$

where

$$[n]_q\equiv\frac{q^n-q^{-n}}{q-q^{-1}}=[n].$$

Thus we can define the q-deformed Laplace operator which is invariant under the $SO_q(3)$ transformation:

$$\Delta=g_{ij}\partial^i\partial^j=g^{ij}\partial_j\partial_i \tag{4.2}$$

where $\partial^i=g^{ij}\partial_j$. The Schrödinger equation in the non-commutative space reads

$$(-\Delta+V)\psi=E\psi \tag{4.3}$$

where V is the interaction potential concerned the physical system.

The commutation relation between ∂^i and x^j can be easily written as

$$\partial^i x^j = g^{ij} + q\check{R}^{-1}{}^{ij}_{kl} x^k \partial^l \tag{4.4}$$

which can be derived from equation (2.11) with $C = q\check{R}$ and the following useful relations:

$$g_{ij}\check{R}^{jk}_{lm} = \check{R}^{-1}{}^{kn}_{il} g_{nm} \tag{4.5}$$

$$\check{R}^{ij}_{kl} g^{lm} = g^{in}\check{R}^{-1}{}^{jm}_{nk}. \tag{4.6}$$

In the previous section we showed that the length square $r^2 = xgx = x^i g_{ij} x^j$, which is invariant under the $SO_q(3)$ transformation, is the centre of the algebra generated by the coordinates x. But this is not true when we consider an extended algebra by including the derivatives ∂ and the differential d together with x. In fact, we have

$$\partial^i r^2 = (1+q^{-1})x^i + q^2 r^2 \partial^i. \tag{4.7}$$

Nevertheless one can show the action of the derivative is still consistent [10]. As in the classical case, we can introduce the positive square root of r^2, denoted by r, by requiring it to have adequate commutation relations with all the other elements of the algebra. Assuming

$$\partial^i = \alpha \frac{x^i}{r} + \beta r \partial^i \tag{4.8}$$

we can get

$$\alpha = q^{-1} \qquad \beta = q \tag{4.9}$$

to reproduce the relation (4.7). In the same way, we have

$$\partial^i r^{-1} = -q^{-2}\frac{x^i}{r^3} + q^{-1} r^{-1} \partial^i \tag{4.10}$$

and in general

$$\partial^i r^n = q^{-1}[[n]]_q r^{n-2} x^i + q^n r^n \partial^i \tag{4.11}$$

where n is an integer and $[[n]]_q$ is defined as

$$[[n]]_q = \frac{1-q^n}{1-q}. \tag{4.12}$$

Defining the q-exponential function as

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[[n]]_q!}. \tag{4.13}$$

We can get the following relation

$$\partial^i \exp_q(-r/a) = -q^{-1}\frac{x^i}{ar}\exp_q(-r/a) + \exp_q(-qr/a)\partial^i \tag{4.14}$$

and

$$\Delta \exp_q(-r/a) = -\frac{q^{-1}+q^{-2}}{a}\frac{1}{r}\exp_q(-r/a) + \frac{1}{a^2}\exp_q(-r/a) \tag{4.15}$$

where a is a constant. The Schrödinger equation for the Coulomb potential $V(r) = -e^2/r$ is

$$\left(-\Delta - \frac{e^2}{r}\right)\psi(x) = E\psi(x) \tag{4.16}$$

where $\psi(x)$ is the eigenfunction and E is the eigenvalue. Through a tedious but straightforward calculation we find the first three solutions to this equation and list them as follows.

(i) Ground level

$$\psi_{100}(x) = \exp(-r/a_1) \qquad a_1 = \frac{q^{-1}+q^{-2}}{e^2} \qquad E_1 = \frac{1}{a_1^2}. \tag{4.17}$$

(ii) The first excited level

$$\begin{aligned}\psi_{200}(x) &= \left\{1 - q^2\left(\frac{r}{a_2}\right)\right\}\exp_q(-r/a_2)\\ \psi_{21m}(x) &= x^m \exp_q(-r/a_2) \qquad m = +, 0, -\end{aligned} \tag{4.18}$$

where $a_2 = a_1 q[2]$, $E_2 = E_1/[2]^2$.

(iii) The second excited level

$$\begin{aligned}\psi_{300}(x) &= \left\{1 - (q^2(1+q^2))\left(\frac{r}{a_3}\right) + \frac{q^6(1+q^2)}{1+q+q^2}\left(\frac{r}{a_3}\right)^2\right\}\exp_q(-r/a_3)\\ \psi_{31}(x) &= x^m\left\{1 - \frac{q^2}{1+q^{-2}}\frac{r}{a_3}\right\}\exp_q(-r/a_3) \qquad m = +, 0, -\\ \psi_{32\mu}(x) &= w_{ij}^{\mu} x^i x^j \exp_q(-r/a_3) \qquad \mu = \pm 2, \pm 1, 0\end{aligned} \tag{4.19}$$

with

$$a_3 = a_1 q^2[3] \qquad E_3 = E_1/[3]^2.$$

The apparent resemblance of the above solutions to the hydrogen atom leads us to christen the corresponding system, the q-hydrogen atom and to guess that the following relations hold in general:

$$E_n = \frac{E_1}{[n]^2} \qquad a_n = a_1 q^{n-1}[n]_q = a_1[[n]]_{q^2}. \tag{4.20}$$

It is easily seen that the q-hydrogen atom has the same degeneracy of the energy levels with its classical counterpart and therefore may have $SO_q(4)$ symmetry correspondingly. In any case it is worthwhile to study this apsect further. As we know, almost only the hydrogen atom and harmonic oscillator system can be solved exactly in quantum mechanics. Now it is proven that they all have their q-analogue solutions [10]. Due to this, the q-Schrödinger equation deserves investigation. In principle, we can discuss the quantum Lorentz group in the same way and construct the q-Dirac equation. This will be reported in a forthcoming paper.

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